Chaotic behaviors of volterra models

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Chaotic Behaviors of Volterra Models

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Abstract

Simulations of three type Volterra models are performed and the trajectories and Lyapunov exponents of the models are calculated for four different initial values which include exact solution of integrable model and in two finite difference methods. The chaotic behaviors are found even in the integrable system with exact solution as the initial values. Lyapunov exponents are shown to depend on the method of difference.

§ 1. Introduction

There are a number of nonlinear dynamical systems which are classified dissipative, conservative, integrable or nonintegrable. The conservative and integrable nonlinear system is known to have the stable soliton-like solutions. At the same time, a lot of instabilities and chaotic behaviors have been observed around them. In the previous paper [1] we have shown that the integrable Volterra model with the zero curvature condition exhibits chaotic behavior for the pulse-like initial values. In the present article, we are going to study the chaotic behaviors of three type Volterra models making use of the exact solution of integrable model and other three cases as the initial values, and for two finite difference methods. In the next section (§ 2), example of chaotic behavior of the discrete equation is given. In § 3, the structure of Volterra model is discussed. In the last section (§ 4), the chaotic trajectories and Lyapunov exponents are numerically obtained for the three Volterra models.

§ 2. Chaotic behavior and discrete system

Dynamical system whose motion $x(t)$ satisfies the equation

$$\frac{dx(t)}{dt}=f(x), \tag{1}$$

is expected to have a unique solution with initial value $x(0)$. On the other hand, the discrete system described by the equation of motion of the form

$$X_{n+1}=f(X_n) \tag{2}$$

often shows a chaotic behavior. A typical example is the logistic model with the equation of motion such as

$$X_{n+1}=f(X_n)=4\,X_n(1-X_n), \tag{3}$$

where X_n belong to I [0,1] . This mapping f shows very complicated orbits depending on the initial value very sensitively. As an indicator of chaotic behavior, the Lyapunov exponent μ, defined by [2]

$$\mu=\lim_{n\to\infty}\frac{1}{n}\log\left|\frac{d(f^n(X_0))}{dX_0}\right|=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\left|\frac{d(f(X_i))}{dX_i}\right|, \tag{4}$$

is often used. If μ is positive and finite for an orbit, we can conclude that it represents a chaotic behavior. Dynamical behavior of nonlinear integrable system in continuous case is described by nonlinear partial differential equations, satisfying the Lax representation [3]

$$\frac{\partial M}{\partial t}-\frac{\partial L}{\partial x}+[M,L]=0, \tag{5}$$

where L and M are evolution and transfer operator respectively, while this takes the form of discrete zero curvature condition

$$\frac{dT_n}{dt}+T_nL_n-L_{n+1}T_n=0, \tag{6}$$

for the discrete nonlinear integrable system, described by the differential-difference equation. In the above eq.(6), the transfer matrix T_n connects the values on the n-th and $n+1$-th sites [4]. In the following, chaotic solutions of several discrete nonlinear system will be studied, in order to check the accuracy of the difference method to study the continuous systems in computer calculus.

§ 3. Volterra models

Let us consider a discrete system described by the following differential-difference equation to be called the Volterra equation

$$\frac{df_n}{dt}=(f_{n+1}-f_{n-1})f_n. \tag{7}$$

In the previous paper we have shown that this nonlinear differential-difference equation satisfies the zero curvature condition [4]

$$\frac{dT_n}{dt}+T_nL_n-L_{n+1}T_n=0. \tag{8}$$

with

$$T_n(\lambda)=\begin{pmatrix} \lambda & f_n \\ -1 & 0 \end{pmatrix}, \tag{9}$$

$$L_n(\lambda)=\begin{pmatrix} f_n & \lambda f_n \\ -\lambda & -\lambda^2+f_{n-1} \end{pmatrix}. \tag{10}$$

This means that the system is integrable. The Volterra equation (7) is very interesting, since it belongs to a class of conservative and integrable dynamical system. It is also related to famous Toda lattice equation which is also integrable [5]. Nevertheless, the solutions of this equation are shown to be chaotic, if the difference method is used to solve it.

§ 4. Simulations of Volterra models with different initial values

As was done in the previous paper, let us consider the following three types of Volterra equations

$$\frac{df_n}{dt}=(f_{n+1}-f_{n-1})f_n, \tag{11}$$

$$\frac{df_n}{dt}=(f_{n+1}-f_{n-1}+1)f_n, \tag{12}$$

and

$$\frac{df_n}{dt}=(f_{n+1}-f_{n-1}+f_n)f_n \tag{13}$$

with the periodic condition

$$f_{n+N}=f_n. \tag{14}$$

The first equation is integrable satisfing the zero curvature condition and this model is called V_1. Similarly we call the second and the third models V_2, V_3. We have performed the numerical integration of these equations, imposing the following initial conditions

a) Exact single soliton solution of V_1 is used [6] [7].

b) Single soliton-like valley is assumed.

c) A trianguler form is assumed.

d) A wave form is assumed.

In all case, a pair of solutions with slightly different initial conditions (the initial values at the mid-point are different by 5%) are calculated. Two types of the finite difference method for the time differential

$$i) \qquad \frac{df_{n,i}}{dt}=(f_{n,i+1}-f_{n,i})/h+O(h) \tag{15}$$

$$ii) \qquad \frac{df_{n,i}}{dt}=(f_{n,i+1}-f_{n,i-1})/2h+O(h^2) \tag{16}$$

are tried, where i is discrete time variable and h is time interval. Using the finite difference method of type i), we obtained the trajectories of the three models V_1, V_2, V_3 in the upper parts of Fig.1 for the initial condition a), where the vertical axis indicates time steps. The corresponding Lyapunov exponents multiplied by n are plotted against time in the lower parts. In Fig.2 similar trajectories and μn of the models for the initial conditions b), in Fig.3 those for the case c) and in Fig.4 those for the case d) are shown. The similar trajectories and μn for the case a), b) are shown in

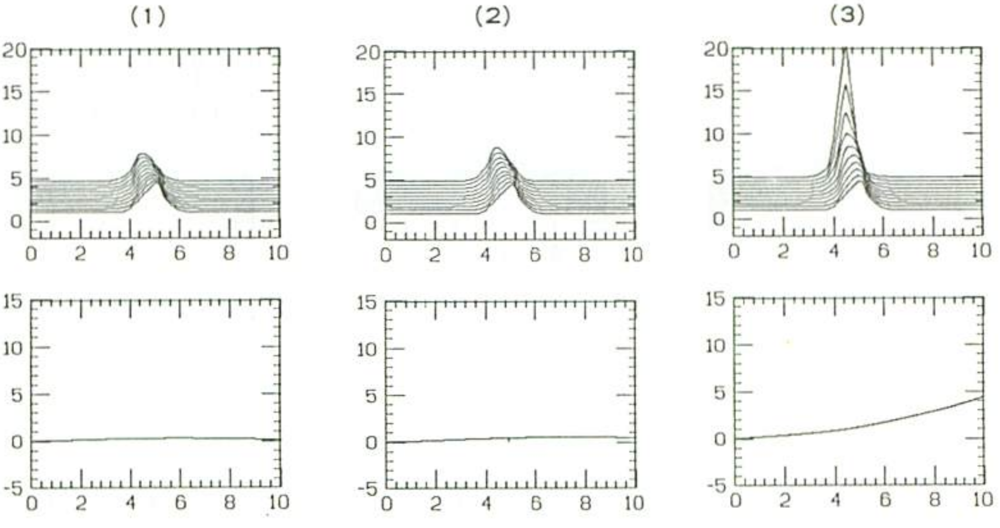

Fig. 1
(1) Trajectories and μn of the model V_1, (2) V_2, (3) V_3 for the initial condition a), using the finite difference method i).

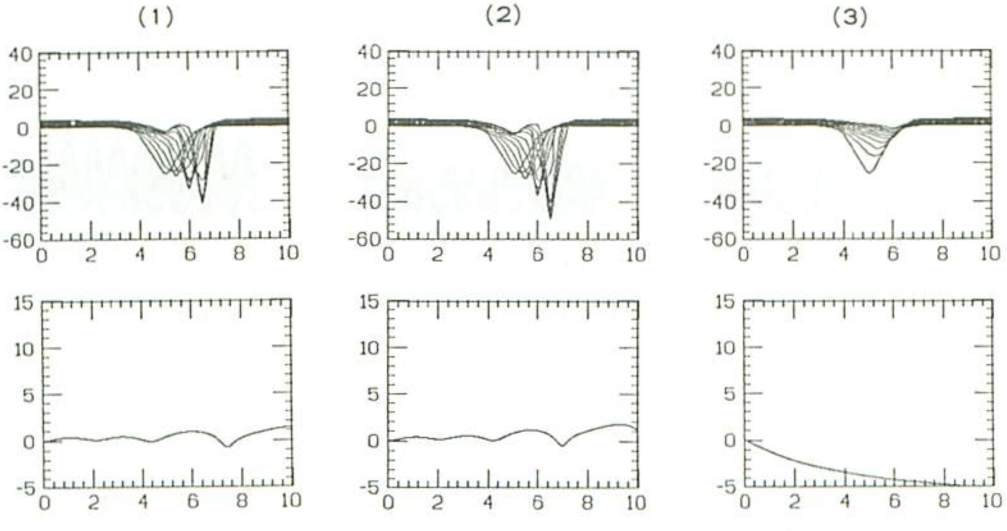

Fig. 2
(1) Trajectories and μn of the model V_1, (2) V_2, (3) V_3 for the initial condition b), using the finite difference method i).

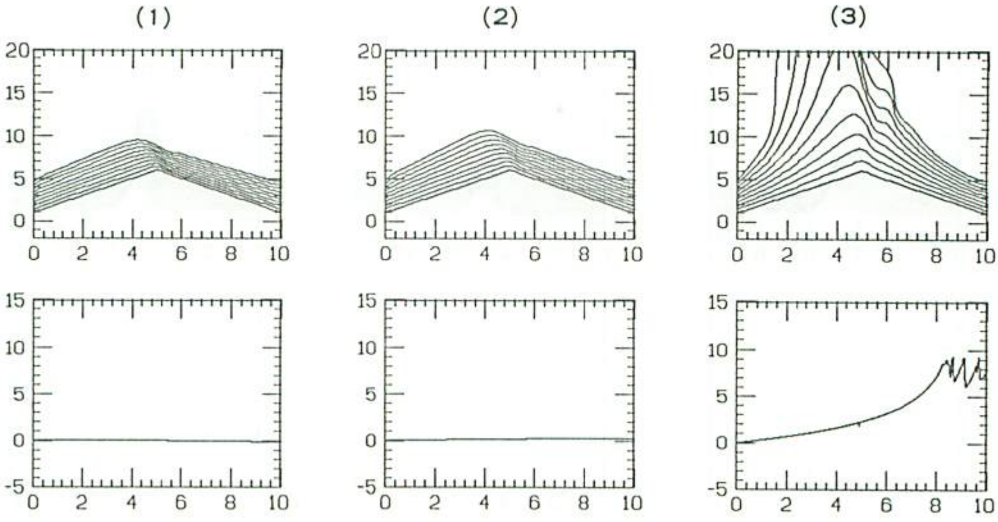

Fig.3
(1) Trajectories and μn of the model V_1, (2) V_2, (3) V_3 for the initial condition c), using the finite difference method i).

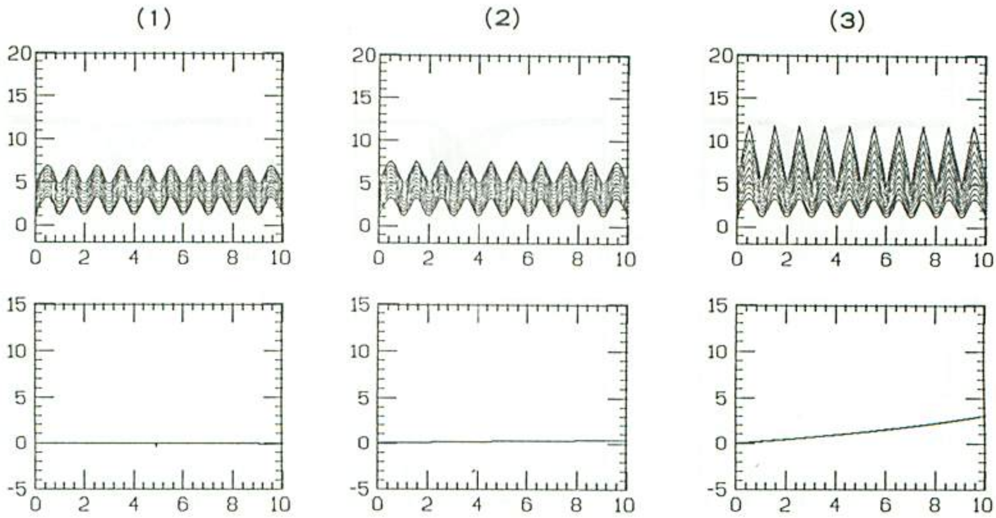

Fig.4
(1) Trajectories and μn of the model V_1, (2) V_2, (3) V_3 for the initial condition d), using the finite difference method i).

Figs. 5 and 6, when another finite difference method ii) is used. At the beginning we have expected that integrable nonlinear model V_1 does not exhibit chaos when the exact solution is used as the the initial values. Of course the model V_1 in the case a) is more stable in contrast to the nonintegrable models such as V_2, V_3, but it still behaves chaotically, especially in Fig. 5 where the finite difference method ii) is used. From the result of present simulations, we may conclude that even the integrable system with exact solution as the initial values will show chaotic behavior. It is to be noted that fluctuations of Lyapunov exponent are induced and sensitively depend on the finite difference method, as is clear from Figs. 5-6.

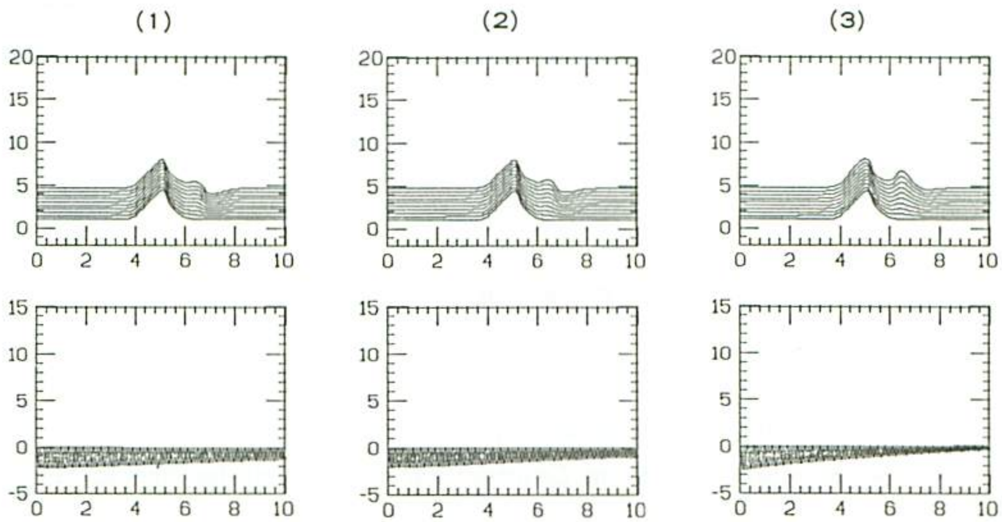

Fig.5

(1) Trajectories and μn of the model V_1, (2) V_2, (3) V_3 for the initial condition a), using the finite difference method ii).

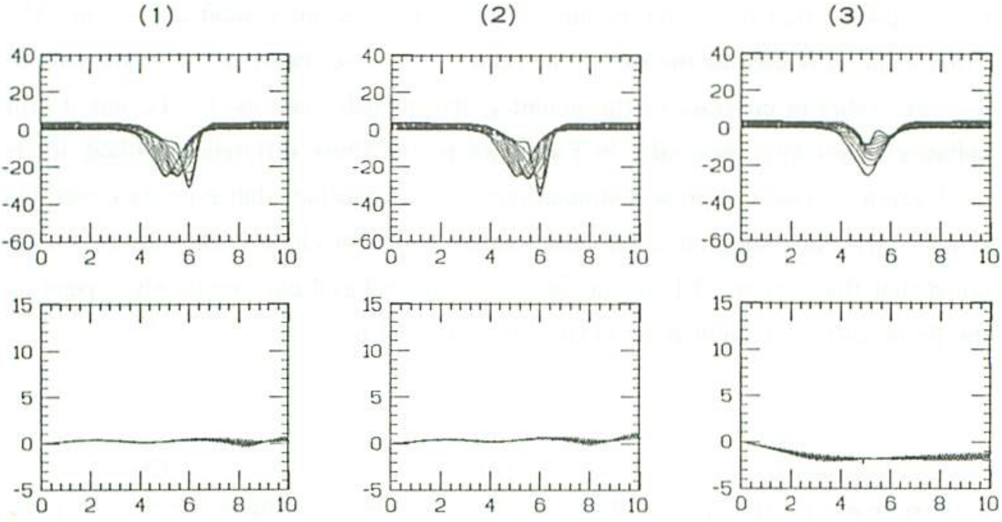

Fig.6
(1) Trajectories and μn of the model V_1, (2) V_2, (3) V_3 for the initial condition b), using the finite difference method ii).

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